Why the Weyl Tile Argument is Wrong

Abstract Weyl (1949) famously argued that if space (or spacetime) were discrete, then Euclidean geometry cannot hold even approximately. Since then, many philosophers have proposed solutions to this argument (e.g., van Bendegem 1987, Forrest 1995). While these authors have successfully suggested some alternative accounts of geometry that allow for discrete space, they have missed an importantly flawed assumption in Weyl's argument: physical geometry is determined by fundamental spacetime structures independently from the dynamical laws. In this paper, I aim to show its falsity through two rigorous examples: random walks in statistical physics and quantum mechanics.

Keywords the Weyl tile argument; discrete space; dynamical laws; rotational invariance; random walk; quantum mechanics.

Space (or spacetime) is called *discrete* just in case it is composed of extended indivisible regions—call them "tiles." Weyl (1949) famously argued that if space were discrete, then Euclidean geometry would not hold even approximately, which would contradict our observations. Therefore, space is not discrete. More specifically, if space were composed of tiles, then the diagonal of a square region would be "equal in length to the side" (Weyl 1949, 43), which would radically violate the Pythagorean theorem. It makes no difference how big the region is: even at the macrolevel, the diagonal of a square region would still be equal to its side. Thus space would not be approximately Euclidean at any scale. As Chen (2021) and others pointed out, this argument relies on the implicit assumption that the distance between any two tiles is equal to the number of tiles between them, which is a standard assumption about discrete space (for example, see Riemann 1866).

Since then, many philosophers have proposed solutions to this argument (e.g., van Bendegem 1987, 1995, Forrest 1995, Chen 2021).[1] While these authors have successfully suggested some alternative accounts of geometry that allow for discrete space, they have missed an importantly flawed assumption in Weyl's argument: physical geometry is determined by fundamental spacetime structures independently from the dynamical laws.[2] In this paper, I aim to show its falsity through two rigorous examples: random walks in statistical physics and quantum mechanics. These examples are intended to be a proof of concept for the claim that physical geometry arises from dynamical laws that do not assume any metric notion ("metric" is a technical term that generalizes the notion of distances). Even if space (or spacetime) is discrete, the right dynamical laws could make spacetime appear approximately Euclidean. This is why Weyl's argument fails.

In the random walk case, I show that on a two-dimensional discrete space represented by $\mathbb{Z}^2$ (pairs of integers), the probability distribution of a wandering tiny cat starting at any given position showing up at each tile is approximately rotationally invariant (it's called "random walk" because, for any time step, the cat randomly walks from a tile to one of its neighboring tiles). If such probabilities are

[1]Note that the Weyl tile argument cannot be solved by simply replacing the square tile arrangement with other simple regular arrangements—Fritz (2012) shows that in order to approximate Euclidean geometry at the large scale, the arrangement of the tiles need to be very irregular and complicated.

[2]While this paper is not concerned with evaluating the solutions proposed by these authors, I can very briefly talk about some of their weaknesses. The solution by Chen (2021), which she argues to be an improvement over Forrest's (1995), relies on a complicated and perhaps contrived account of discrete space (in her account, real-valued primitive distances are held by a vast number of "neighboring" tiles). van Bendegem (1995) discussed and criticized his own (1987) proposal, but his new proposal has a complicated ideology: he posited both a set of microscopic geometry entities ("*t*-point," "*t*-line", etc.) and macroscopic ones ("point," "line," etc.), and the latter are governed by principles not reducible to those of the former.

the only observable quantities, then we have an embedding map from the discrete space to Euclidean space that approximately preserves all structures and observations. This means that Euclidean geometry is approximately recovered at the empirical level.

In the case of quantum mechanics, I show that for a quantum mechanical system starting with a sufficiently spread-out position wavefunction, the amplitude of it reaching each tile at a later time is approximately rotationally invariant. Assuming that the probability of a quantum mechanical system in a region at a time (which is the square of the amplitude) is the only kind of observational quantity, this again means that Euclidean geometry is approximately recovered on the observational level. the random walk case is chosen for its rigor and conceptual clarity while the case of quantum mechanics adds more physical relevance.

Based on these cases, I criticize two "geometricist" assumptions underlying Weyl's argument (*geometricism* is roughly the view that geometry is more fundamental than dynamics; the opposite view is called *dynamicism*): (1) large-scale or observable physical geometry is determined by fundamental spacetime structures independent of dynamical laws; (2) some geometric structures (including the metric structure) are ontologically and explanatorily prior to dynamics and must be presumed by the latter. Against (1), in the two cases, the dynamical laws play an essential role in determining observational geometry. Against (2), no metric notion is presupposed by the dynamical laws. It is also important to note that, while the two cases are simple toy examples, they are not contrived or ad-hoc, and therefore suggest a natural possibility that our actual fundamental laws similarly give rise to Euclidean geometry on the ordinary scales. Thus, we should reject the geometricist assumptions in our understanding of spacetime. While this paper focuses on criticizing Weyl's argument and thereby defending the possibility of discrete space (or spacetime), it also seeks to draw a general lesson about physical geometry by rejecting the geometricist assumptions. (For more discussions on dynamicism, see for example Brown 2005, Norton 2008, [Anonymized a])

The paper (in particular, Section 2 and 3) is technical in nature, but the main text aims to be generally intelligible for interested philosophers while the appendices may require more technical fluency.

1 Weyl's Tile Space

The Weyl tile argument against discrete space implicitly assumes a simple *counting* account of distance in the tile space (Chen 2021). To explain, we first need to define a topology on the tile space through the primitive notion of *connectedness*, which is an irreflexive and symmetric binary relation (see Roeper 1995 for a topological framework based on "connectness"). For example, for the two-dimensional discrete space, we can postulate the following topology:

> Connectedness. Let the tiles be represented by members of $\mathbb{Z}^2$. For any tiles x and y, they are *connected* iff $|x-y|=1$.[3]

Informally, this says that every tile is connected to its neighboring four tiles (i.e., its left, right, up, and down tiles).[4] Then, we can obtain distances between tiles by counting connected tiles between them:

> Distance-by-counting. For any tiles A, B, if $C_1, C_2, \ldots C_n$ are the least number of tiles that are pairwise connected and connected to A, B, then the distance between A, B is $n+1$.

Distance-by-counting is the discrete version of the standard path-dependent account of distance, namely that the distance between two points equals the length

[3] For any $x=(z_1,z_2)\in\mathbb{Z}^n$, $|x|=\sqrt{z_1^2+z_2^2}$.

[4] Note that in Weyl's argument, each tile is connected to eight neighboring tiles rather than four. But the difference does not affect the essence of Weyl's argument.

of a shortest path (or extremal ones in the case of spacetime). In the discrete case, a path is composed of connected tiles and its length equals the number of those tiles. Distance-by-counting is very intuitive and natural for discrete space and has indeed been endorsed by many, including Riemann in his foundational work (1866) for differential geometry. Apriorily, this seems to be the best account due to its simplicity and elegance (see Forrest 1995).

But the appearance is wrong, because Distance-by-counting falsely assumes that physical geometry and the distances we observe can be determined by fundamental spatial structures independently from dynamical laws that govern how matters behave and interact. As I will argue, this assumption is unwarranted. (Note that the notion of distance in Distance-by-counting and the geometry it determines should be empirical, that is, being observable and measurable by devices like rigid rods and light rays and clocks in the case of spacetime, since a non-empirical geometry could not contradict our observations as the argument goes.)

In the upcoming sections, I will offer two rigorous cases in which the underlying spatial structure is simple as Weyl imagined (minus any metric structure), but we can nevertheless observe approximately Euclidean geometry. I will focus on the recovery or emergence of approximate isotropy (or rotational invariance) from the tile space, since this removes the main barrier for recovering of Euclidean geometry (e.g., see Forrest's (1995) discussion on the anisotropy problem). Also, like other authors, the recovery of approximate Euclidean geometry is treated as approximate embeddability of physical quantities into Euclidean space (e.g., see Chen 2021, Appendix A).[5]

[5] The difference is that Chen and many other authors in the literature seek to embed distances to Euclidean space as part of the fundamental physical quantities, while my project in the rest of the paper is restricted to embedding all the observables of a given dynamics—albeit very rudimentary ones—to Euclidean space. The idea is that large-scale distances as empirical quantities supervene on these observables.

2 Case One: Random Walk

In this section, I will present the toy example of a random walk to illustrate how Euclidean geometry can emerge from the dynamics that does not presuppose it. Random walks are studied in statistical physics, which is a branch of physics that studies how properties of macroscopic systems arise from stochastic microscopic motions. As a proof of concept, this example is chosen for its mathematical simplicity and intuitiveness as well as conceptual rigor, even though it does not depict a physically realistic situation. The sketched proof in the main text should be accessible to interested general philosophers, while most mathematical details of the proof are left to Appendix A for more technically informed readers.

Following Weyl, I will focus on the two-dimensional square tile space for simplicity. Imagine that there is a tiny cat running from tile to tile. From each tile, the cat can move along four directions, characterized by two basis vectors $\pm e_1, \pm e_2$ (with $|e_i|=1$ for all i).[6] This can also be captured by the topology postulated in the previous section (namely that each tile is connected to four neighboring tiles). There is no metric structure on top of this. But I will define the *Euclidean distance* between two tiles $x, y \in \mathbb{Z}^2$ to be $|x-y|$. This is *not* a postulation of a geometric structure, as I will emphasize more later. We can prove the following:

> Isotropy. For any starting position x, and for any two tiles that have approximately the same Euclidean distance to x, the probabilities of the cat showing up in them, if nonzero, are approximately the same after sufficiently long time (namely after the cat has taken sufficiently many steps).[7]

[6] For any $x=(z_1,z_2)\in\mathbb{Z}^n$, $|x|=\sqrt{z_1^2+z_2^2}$.

[7] The qualification of "non-zero" is imposed because for any number of steps, the probability of the cat showing up in any tile with the opposite parity is zero. For example, after any even number of steps, the probability of the cat in any tile represented by (x,y) with $x+y=odd$ is zero.

This result is similar to a central theorem in probability theory, *the central limit theorem* (CLT), ac-

Proof Let the starting position of the cat be the origin $(0,0)$. It's easy to see that, after the first step, the probability of the cat showing up in each of the four neighboring tiles is $1/4$. Our goal is to calculate the probability for any tile after sufficiently many steps. If the cat could not backtrack and only move in two directions, then such a probability would be easy to calculate. In that case, to reach (x,y) after n steps, the cat needs to take a total of x steps in the "horizontal" direction and a total of y steps in the "vertical" direction. Using basic combinatorics, the result would be $(1/2)^n\binom{n}{x}$, since each path the cat might take has a probability of $(1/2)^n$ and there are $\binom{n}{x}$ possible paths. The problem is trickier now that the cat can move back and forth along horizontal and vertical directions.[8] Fortunately, with a clever trick, we can obtain that the probability of reaching (x,y) after n steps, if nonzero (i.e., when $x+y$ is of the opposite parity than n):

$$(1/4)^n\binom{n}{(n+x+y)/2}\binom{n}{(n+x-y)/2} \tag{1}$$

See Appendix A.1 for the detailed proof.

Furthermore, it follows from a result in Gallager (1968) that (1) is approximately equal to the following when $x \ll n, y \ll n$ (see Appendix A.2 for the proof):

$$\frac{2e^{(x^2+y^2)/n}}{\pi\sqrt{n^2-(x^2+y^2)}} \tag{2}$$

cording to which the normalized sum of independent random variables (under certain conditions) tends towards a normal distribution (see van der Vaart 1998). We can apply this theorem to our case of a random walk. CLT implies that as time tends to infinity, the distribution of the probability converges to that of the n-dimensional normal distribution centered around the starting position. More specifically, after n steps, the probability of the cat showing up at a region around a particular tile with the size $O(\sqrt{n})$ is rotationally invariant as n tends to infinity. Isotropy is stronger than this general result because it is about the approximate isotropy of the probability distribution on single tiles rather than large regions.

[8] The obvious idea in the simple case would not work. For example, let the number of steps along the four directions be a,b,c,d respectively. The probability of reaching (x,y) after n steps is equal to $(1/4)^n\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$, with $a+b=x$ and $c+d=y$. But this probability cannot be calculated because there are too many unknown variables.

The result (2) only contains the Euclidean form x^2+y^2 and is therefore rotationally invariant. This can be straightforwardly generalized to any tile as the starting position (as it is essentially a matter of coordinate translation). This concludes the proof for Isotropy. □

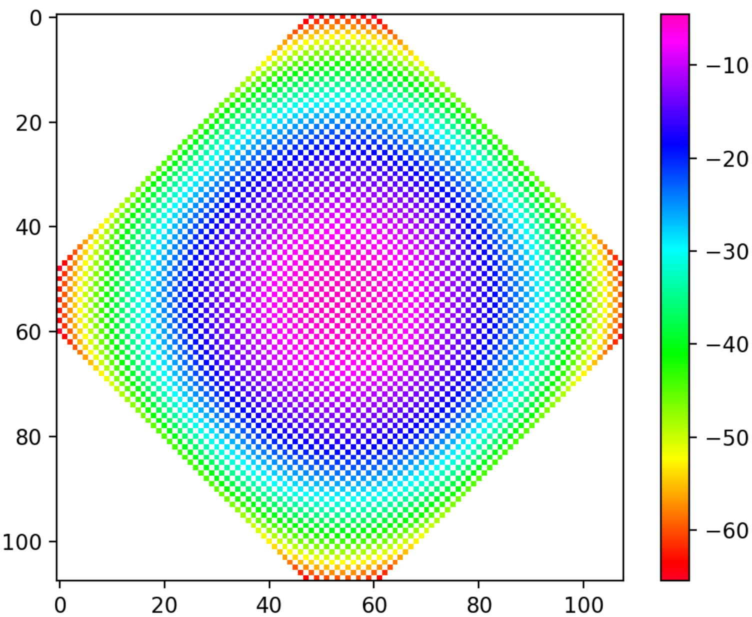

Figure 1: the probability distribution of the cat at t=60 on the tile space. Note that the colors correspond to the *logarithm* of the probabilities in order to differentiate between small probabilities.

Now that the proof is given, let me further compute and visualize the probability distribution of the cat at a time to provide an intuitive picture of what it is like. In Figure 1, I show the probability distribution of the cat at t=60 as an example.[9] We can see that it is indeed approximately rotationally invariant: the significant probabilities represented by purple and deep blue colors form circle-like shapes. Note that the white areas are tiles that the cat cannot reach at t=60, and we can see that all the tiles with the opposite parity of t are indeed white.

It follows from Isotropy that there is an embedding from the tile space into Euclidean space that approximately preserves the probability distribution of the

[9]All the codes that produce the figures in this paper are accessible upon request or at [Anonymized webpage].

cat after sufficiently long time (or more strictly speaking, from regions of the tile space of sizes greater than one to regions of Euclidean space so that the effects of white squares can average out). Assuming that such probabilities are our only observations, this means that our observations would be about the same as in Euclidean space. In this sense, the apparent Euclidean geometry emerges from the tile space under the dynamics of a random walk.

I would like to highlight some important features of this example. First, in this example, there is no isotropic (or approximately isotropic) spacetime structure existing prior to dynamics. Indeed, I haven't defined a metric structure, and the topology is far from isotropic. It is worth emphasizing that the notion of "Euclidean distance" is just a feature of our representation of the tile space, which allows us to conveniently specify the embedding just mentioned. Emphatically, it does not refer to any primitive geometric structure involved in the dynamical laws. Second, the dynamical law posited in this example is not contrived or overly complicated. Indeed, it is the simplest one studied in statistical field theory. The simplicity of laws is an important feature that distinguishes this case from the simulation of continuous mechanics on discrete pixels in computer programs. In such programs, the description of (say) a rotation either makes use of Euclidean geometry in which the discrete pixels are embedded or gets very messy. This messiness or complexity is one main reason why we do not consider reality as composed of pixels on which continuous motions are simulated. But we should reconsider that in light of the current example.

Also, notice that the recovery of isotropy in this case does not critically involve macroscopic regions consisting of many tiles. Rather, the statistical correlations between individual tiles are already isotropic. This makes the case radically different from—for example—Bendegem's (1987, 1995) proposals, which crucially rely on macroscopic geometric entities for recovering Euclidean geometry (see Footnote 2 for a brief explanation of some costs of his proposals).

Of course, the random walk case is physically unrealistic: it does not depict the behavior of any fundamental particle or field; our space (or spacetime) is not two dimensional; actual observations are vastly more complicated, to name a few. The physical situations most relevant to this case are those studied by statistical field theory, such as the transmission of heat (the transmission of heat would be isotropic even if the molecules involved moved along discrete tiles). But this is not a realistic interpretation of the current example, since the molecular movements occur at much larger scales than the fundamental unit of space.

Nonetheless, this case already constitutes serious evidence against the implicit assumption in the Weyl tile argument that the empirically observable distances are determined by counting the number of tiles. A dynamical law is *isotropic* (or *rotationally invariant* or has *rotational symmetry*) if it is not affected by any rotation of the system it applies to (mathematically, we can say that the application of the law to a system *commutes* with any rotation of the system). Isotropy shows that the simple law of a random walk is isotropic with regards to a single cat (and it is not difficult to check that this is the case for any number of cats), despite the underlying space being discrete and not isotropic. It is also possible that our fundamental laws are isotropic even if spacetime is discrete. Therefore, it is a serious problem for Weyl's argument to exclude this possibility. The symmetries of the dynamical laws, which determine observational symmetries, may not be among the symmetries of spacetime.

I intend this case to do the conceptual heavy-lifting for my objection to the Weyl tile argument due to its conceptual clarity and simplicity, but I will now turn to the case of quantum mechanics for a more physically relevant example.

3 Case Two: Quantum Mechanics

In this section I will show how observational rotational invariance can emerge from quantum mechanical systems in discrete space that does not presuppose any rotationally invariant geometric structure. The sketched proof in the main text should be accessible to philosophers with some conceptual familiarity with the formalism of quantum mechanics, while more technical details are left to Appendix B. Despite this case being more complicated and subtle than the random walk case, its relevance to fundamental physics can bolster the sentiment that it is *really* possible that our dynamical laws are written on discrete space (or spacetime) and are approximately isotropic for all we know.

For generality, I will consider an n-dimensional tile space. Let each tile be represented by a member of $\mathbb{Z}^n$. From each tile, there are $2n$ directions characterized by n basis vectors $\pm e_1, \pm e_2, ..., \pm e_n$ ($|e_i| = 1$ for all i). I will show that the following claim is true:

> Isotropy-qm. For any quantum mechanical system with its initial position spread out in a sufficiently large region $A \subset \mathbb{Z}^n$, its time evolution is isotropic: it does not make any significant physical difference whether we rotate the initial position or apply a similar rotation to it at a later time.[10]

As a special case, if the starting position A is spread out and rotationally invariant, then its evolution will continue to be rotationally invariant: for any $y, z \in \mathbb{Z}^n$ that have approximately the same Euclidean distance to (the center of) A, the amplitudes of the system at y and z at any time are approximately the same.[11] This

[10]"Spread out" means that the position wavefunction of the system does not vary much in short distances. Very roughly, a rotation of a system relative to position $a \in \mathbb{Z}^n$ as a wavefunction maps its values at every $x \in \mathbb{Z}^n$ to $y \in \mathbb{Z}^n$ that has similar Euclidean distance to a as x. A more rigorous, mathematical description of rotation can be found in Appendix B.

[11]A center of a region is a tile that minimizes the maximal Euclidean distance to other tiles in the

implies that for any two regions that consist of tiles with similar Euclidean distances to the starting region, the probabilities of observing the system in those regions are approximately the same. Assuming that such probabilities are our only observables, there is an embedding from the tile space to Euclidean space that preserves all structures and observations, as in the previous case. Notice that Isotropy-qm requires the initial position to be spread out, which is why this result is conceptually weaker than Isotropy.[12] But this does not necessarily mean it is less satisfactory, as it is possible that we can only ever observe or prepare a system spread out in a suitably large region.

Before I go into the proof, I want to forestall a dismissive reaction. Some people (e.g., from physics communities) may find Isotropy-qm obvious or even a stronger version of it that does not require the initial position to be spread out obviously true, and therefore think there is no need for a demonstration.[13] However, this reaction is unjustified. To show this, I demonstrate in Appendix B.3 that a stronger claim (which some consider true) is actually false: for any quantum mechanical system with initial position $x \in \mathbb{Z}^n$, its wavefunction will not evolve to be approximately rotationally invariant for any significant period of time. For some readers, this negative result might be the most interesting one. Other readers might also find it useful to have a negative claim to contrast the two cases of the paper with.

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region. Note that, as explained earlier, "Euclidean distance" is just a convenient tool for specifying how we can embed the tile space into Euclidean space given a representation of the tiles by $\mathbb{Z}^n$. It does not refer to a primitive geometric structure of space.

[12]Another difference between this case and the first one, as we will see, is that the time is not discrete in this case. Unlike the first case, for any time $t > 0$, the position wavefunction is nowhere zero. Heuristically, we can think of continuous time as composed of infinitely many infinitesimal durations, and therefore for any finite time, the "Schrödinger's cat" could show up anywhere.

[13]Why might some people think so? I cannot hope to explain in a way that does justice to all those who have such beliefs, but here's a rough attempt. The gist is that momentum wavefunction is approximately isotropic for small momenta. Small momenta correspond to large distances. Thus, focusing only on large distances, the position wavefunction is approximately isotropic. As we will see, this reasoning has some semblance of my proof, but is fallacious.

Proof To prove Isotropy-qm, we shall start from the Schrödinger equation, which governs the evolution of quantum mechanical systems (setting the Planck constant $\hbar$ to one):

$$i\frac{d}{dt}\Psi(t) = \hat{H}\Psi(t), \tag{3}$$

where $\Psi(t)$ is the position wavefunction of the system at t, which assigns a complex-valued amplitude to each spatial point, and $\hat{H}$ is the Hamiltonian operator on the wavefunction (which indicates the total energy of the system). In order to apply the Schrödinger equation to discrete space, we need to formulate the discrete version of the equation. Since I will exclusively consider the discrete case, I will use the same notation without risking ambiguity. First, the discrete version of the wavefunction $\Psi(t)$ for any given t can be considered a complex-valued function over the tile space:

$$\Psi(t) : \mathbb{Z}^n \to \mathbb{C}.$$

For the right side of (3), we need to discretize the Hamiltonian. If we set the mass of the system under consideration to one and ignore its potential energy, the Hamiltonian is equal to its kinetic energy $-\frac{1}{2}\sum_i(\frac{\partial}{\partial x^i})^2$. A natural discrete definition of $\frac{\partial}{\partial x^i}$ would be the difference of the value of a given function between neighboring tiles along a certain direction. That is:

$$\frac{\partial}{\partial x^i}\Psi(t,x) = \Psi(t,x+e_i) - \Psi(t,x), \text{ or} \tag{4}$$

$$= \Psi(t,x) - \Psi(t,x-e_i) \tag{5}$$

Then, the discrete version of the Hamiltonian is this (concerning only its kinetic energy part):[14]

$$\hat{H}\Psi(t,x) = -\frac{1}{2}\sum_i (\Psi(t,x+e_i) + \Psi(t,x-e_i) - 2\Psi(t,x)) \tag{6}$$

$\Psi(t)$ is a function in the position space, and it is useful to transform it into one in the momentum space, where we can prove rotational invariance more easily. The inverse Fourier series of $\Psi(t)$ is its momentum space counterpart $\tilde{\Psi}(t): \mathbb{R}^n \to \mathbb{C}$:[15]

$$\tilde{\Psi}(t,p) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi i p x}\Psi(t,x) \tag{7}$$

where "*px*" refers to the inner product of n-vectors p and x. Notice that this momentum wavefunction is periodical. That is, $\tilde{\Psi}(t,p) = \tilde{\Psi}(t,p+e_i)$.[16] Therefore, we can consider $\tilde{\Psi}(t)$ as a complex-valued function defined on the quotient space $\mathbb{R}^n/\mathbb{Z}^n$, which means that we can "collapse" all real space points with integer distances away into one point. This domain can be represented by the unit square around the origin: $B = [-1/2, 1/2] \times [-1/2, 1/2]$ ("B" stands for "Brillouin zone"). Then we have $\tilde{\Psi}(t): B \to \mathbb{C}$. We can transform it back to the position wavefunction

[14]This is obtained by applying $\frac{\partial}{\partial x^i}$ twice in opposite directions (that is, (4) and (5) respectively). If we only apply (4) twice, then we would have

$$\hat{H}\Psi(t,x) = -\frac{1}{2}\sum_i (\Psi(t,x+2e_i) - 2\Psi(t,x+e_i) + \Psi(t,x))$$

which is also a legitimate choice, but makes calculation more complicated.

[15]$\Psi(t)$ is the *Fourier series* of its momentum counterpart $\tilde{\Psi}(t)$ because the domain of $\Psi(t)$ is discrete. In (7), we can see that the momentum space wavefunction is the discrete sum of a series.

[16]Here's the derivation (omitting t for brevity):

$$\tilde{\Psi}(p+e_i) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi i(p+e_i)x}\Psi(x) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi i p x}e^{-2\pi i e_i}\Psi(x) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi i p x}\Psi(x) = \tilde{\Psi}(p)$$

in the following way:

$$\Psi(t,x) = \int_{p\in B} e^{2\pi i p x}\tilde{\Psi}(t,p)dp \tag{8}$$

It follows that the time evolution of Ψ is approximately isotropic if the evolution of $\tilde{\Psi}$ is approximately isotropic (see Appendix B.2 for the proof). Moreover, we can show that, assuming that the initial position wavefunction $\Psi(0)$ is sufficiently spread out, the time evolution of $\tilde{\Psi}$ is indeed approximately isotropic (see Appendix B.1 for the proof). Isotropy-qm follows. □

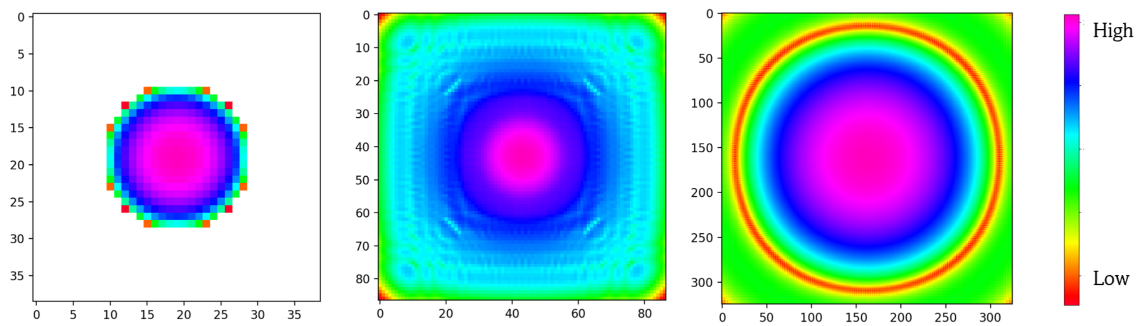

Figure 2: The evolution of a wavefunction with initial radius of 10. All significant parts of the wavefunction are plotted. (Left) $t = 0$; (Middle) $t = 30$; (Right) $t = 300$. Note that the scales are different between plots, since the wavefunction is more spread out as time passes; also, the values represented by the same colors are different between plots, since the amplitudes generally get much lower as the wavefunction is spread thin.

Like the random walk case, I shall corroborate the proof by computing and visualizing the time evolution of the position wavefunction for a more intuitive picture. We start with an approximately rotationally invariant wavefunction that is spread over tiles represented by (x,y) with $x^2 + y^2 < 10^2$, and compute its evolution over time. In Figure 2, I have plotted the wavefunction at $t = 0, t = 30, t = 300$ respectively. We can observe that in all these times, the wavefunction is indeed approximately isotropic: the significant parts of the wavefunction represented by purple and dark blue blocks at these times have circle-like shapes. But we also observe that the isotropy gets more perfect as time passes. In early times such as

$t = 30$, we can still observe vertical and horizontal stripes in blue and green, which are deference patterns. These almost disappear in later times such as $t = 300$: we can see that the deference stripes in red, yellow and green are almost perfect circles. (Note that the color and size scales are different between plots; see the explanation in the caption.) In contrast, in Appendix B.3 we can see that the evolution of a wavefunction with its initial position at a single tile looks very different and is never rotationally invariant. This stark contrast might surprise some readers given that the spread-out wavefunction simply "sums over" copies of the other.

Does the isotropic time evolution of quantum mechanical systems mean the emergence of Euclidean geometry at the empirical level? Strictly speaking, observable Euclidean geometry is not yet recovered since the apparent distances are not even defined. But this is not a problem in principle: all matters, including length-measuring devices, are reducible to quantum mechanical systems (or quantum fields according to quantum field theory, or something more fundamental according to future physical theories). Thus, ensuring that the observable quantities in the fundamental theory are isotropic also ensures that the apparent distances will be Euclidean.

While this case has more physical relevance than the random walk case, this is still a very simplistic example. For one thing, quantum mechanics is not a fundamental theory due to its conflict with relativity (in this example, the temporal dimension is still pre-relativistically separate from the spatial dimensions and is not discretized). Quantum field theories are more fundamental, and there is indeed a sentiment in the community that a large-scale isotropy will emerge from fundamental laws defined over discrete spacetime.[17] But to convert such a sentiment into rigorous frameworks, theorems, and proofs, there is still a lot to be done

[17] This is a common hope in the community of lattice quantum field theory (LQFT). Note that the lattice involved in LQFT is not necessarily intended to be read realistically, but as a computational device. (For example, see Montvay and Munster 1994)

and nothing certain can be said at this stage. Furthermore, even quantum field theories are not sufficiently realistically for our purposes, since they do not address the problem of quantum gravity (that is, incorporating the gravitational field into the frameworks of quantum field theory). The successful handling of quantum gravity is important to a realistic story concerning the fundamental structure of spacetime.[18]

4 Against Two Tenets of Geometricism

So far I have laid out the main results and my objection to Weyl's argument, but it is worth expounding on where the argument went wrong. In particular, I will criticize two (interrelated) assumptions underlying it: (1) large-scale or observable physical geometry is determined by fundamental spacetime structures independent of dynamical laws; (2) some geometric structures (including the metric structure) are ontologically and explanatorily prior to dynamics and must be presumed by the latter. I shall call them "the two tenets" of *geometricism*, the view that geometry is more fundamental than dynamics, with the opposite view called *dynamicism*.[19] (The debate between the two positions usually proceeds in the con-

[18]In [Anonymized b], I argue that we do not have a realistic understanding of spacetime even at the experimentally accessible levels in the absence of a successful theory of quantum gravity.

The research programs of quantum gravity that appeal to discrete spacetime are conceptually similar to the project of this paper, only with vastly more complexity: they involve developing physics on discrete spacetime that yields general relativity as its limit. See for example Hamber (2009).

[19]Of course, I do not intend the two tenets to do justice to all variants of geometricism. For example, some geometricists (e.g., Maudlin 1988, 2012) consider the metric structure as an essential feature of spacetime, while others (such as Earman and Norton 1987, Norton 2008) may think that spacetime is represented by metrically amorphous manifolds and the metric is a matter field in spacetime. But I do believe that the two tenets are commonly held by geometricists as well as those who haven't entered the debate. This is understandable. Classical dynamical laws often presuppose geometrical notions. For example, the law of inertia in Newtonian mechanics says that a free system moves along a straight line, where straightness is a geometrical notion. In relativistic theories, an analogous law that says that a free falling system moves along a time-like geodesic. Thus, it is natural to think that spacetime has a metric structure that dictates how matter in spacetime behaves.

tinuum case, but I think the discrete case examined in this paper can be more helpful in clarifying how dynamics can be more fundamental than geometry, and how we can have interesting physics without any fundamental metric structure.)

First, the Weyl tile argument is wrong about how large-scale distances emerge from the fundamental structures. As pointed out in Section 1, the argument relies on Distance-by-counting, which assumes that distances and in general the physical geometry can be determined independently from the dynamical laws. In the two examples discussed in the paper, I have shown that dynamics play a crucial role in determining physical geometry. The observable isotropy is determined by the dynamical laws that govern the movement of the tiny cat in the random walk case and the evolution of the wavefunction in the case of quantum mechanics. In general, since we observe physical geometry with various measuring devices like rigid rods, it is natural to expect that the measurement we get is partly determined by how those devices work. (Note that this is a hotly debated claim in the debate between geometricism and dynamicism; see Brown 2005, Maudlin 2012, Norton 2008, Menon 2019, [Anonymized a].)

Note that in the two cases, even if we posit a metric structure according to Distance-by-Counting, or in any other ways, it would have no empirical consequences, since it does not play any role in the dynamical laws. So we should not posit such a structure, given that we should not posit structures with no empirical consequences.

Second, it is a mistake to assume that geometry must be presumed by dynamics. Hopefully this is already clear from the previous discussion, but is still worth emphasizing. The difference between this tenet of geometricism and the first one is this: one may grant that the apparent geometry arises from dynamical laws but still insist that some fundamental geometric structure must be presumed by dynamics. For example, in the famous Poincaré disk scenario (Poincaré 1912[2018]), the geometry appears hyperbolic to the residents of the disk because of the dynamics (there is a universal force that shrinks rigid rods and bends light beams). But we know by stipulation that such dynamics laws are still defined over Euclidean geometry, which exists fundamentally. Thus the fact that dynamical laws play a role in determining apparent geometry does not necessarily mean there is no underlying "real" geometry. It is a common assumption that dynamics laws need to be "written on" spacetime geometry (Earman 1989, 46). But the two cases considered have demonstrated that the fundamental dynamical laws do not need a prior metric structure. The discrete version of the Schrödinger equation only requires the topological structure and the derived "differential" structure.

One may object that the topological structure or just the set of tiles itself should count as a geometrical structure, and therefore I haven't refuted the second tenet. Indeed, Norton (2008) objects to dynamicism by arguing that spacetime coincidence is not derivable from dynamical laws but must be presupposed. Fair enough—in this paper, I only intend to argue that no metric structure needs to be presupposed by the dynamics. To get rid of spacetime altogether, we may need a very different framework such as *algebraicism* rather than the standard point-set-theoretic framework, which I will not delve into here (e.g., see Geroch 1972, Connes 2013, Menon 2019, Chen and Fritz 2021).[20]

5 Conclusion

The Weyl tile argument against discrete space implicitly assumes that the symmetries of geometry help determine symmetries of matter systems: since the geometry of the tile space is not rotationally invariant, the physical laws and observables

[20]These authors did not discuss the discrete case explicitly. But the discrete case can be a special case of the formalism proposed in Chen and Fritz (2021). I shall discuss this elsewhere, because algebraicism is a topic all on its own.

would also lack rotational invariance. However, I have shown that the observable physical states in the random walk case and quantum mechanics are decoupled from the tile geometry. Even though the tile space is radically non-Euclidean in the sense that the diagonal of a square contains twice as many steps as the side, the observable geometry can still be approximately Euclidean.

A Random Walk

Theorem A.1 *The probability of reaching* (x, y) *from* $(0,0)$ *after* n *steps is equal to zero (when* $x+y$ *has the opposite parity of* n*) or*

$$(1/4)^n \binom{n}{(n+x+y)/2} \binom{n}{(n+x-y)/2} \tag{9}$$

Proof It should be obvious that the probability of reaching (x, y) from $(0,0)$ after n steps is equal to zero when $x+y$ has the opposite parity of n, so in what follows, I will assume $x+y$ has the same parity as n. We imagine that in every step the cat takes, the cat actually takes two half-steps along the diagonal directions. We assume that there are four possible ways the cat can move in one step from position 0: $(1/2, 1/2)+(1/2, -1/2)$, $(1/2, 1/2)+(-1/2, 1/2)$, $(-1/2, -1/2)+(1/2, -1/2)$, $(-1/2, -1/2)+(-1/2, 1/2)$. This corresponds to the four possible ways to move in the original situation $(\pm 1, \pm 1)$. We can consider the original situation and the imagined situation as two ways of representing the same physical situation. As an analogy, in chess, a knight's move can be equivalently considered as one L-shaped step or as consisting of first moving one row (or file) and then moving two files (or rows). Let's call the number of steps that the cat takes in the aforementioned four possible ways respectively ac, ad, bc, bd (Figure 3). Because the number of steps involving a (i.e., $ac+ad$) and the number of steps involving c (i.e., $ac+bc$) are independent, the probability of reaching (x, y) after n steps is equal to

$$(1/2)^n \binom{n}{ac+ad} \cdot (1/2)^n \binom{n}{ac+bc}$$

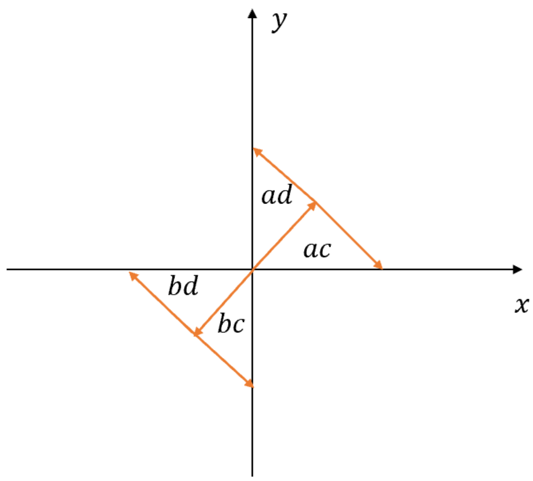

Figure 3

with $ac + ad + bc + bd = n$ and $ac - bd = x, ad - bc = y$. It follows that the probability of reaching (x, y) after n steps is equal to

$$(1/4)^n \binom{n}{(n+x+y)/2} \binom{n}{(n+x-y)/2}$$

Theorem A.2 *Assuming $x \ll n$ and $y \ll n$, (9) is approximately equal to*

$$\frac{2e^{(x^2+y^2)/n}}{\pi\sqrt{n^2-(x^2+y^2)}} \tag{10}$$

Proof Assuming both n and $n-k$ are very large, we have the following theorem (Gallager 1968):

$$\binom{n}{k} \to \sqrt{\frac{n}{2\pi k(n-k)}} e^{nh(k/n)} \tag{11}$$

where $k \in [1, n-1] \cap \mathbb{Z}$ and $h = -x \ln x - (1-x)\ln(1-x)$. Plugging (9) into (11), the result would be of the form αe^{β}, where

$$\alpha = (1/4)^n \sqrt{\frac{4n^2}{\pi^2(n^4 - n^2(x^2+y^2) + (x^2-y^2)^2)}} \tag{12}$$

Since $x \ll n$ and $y \ll n$, we have $(x^2-y^2)^2 \ll n^2$, and thus (12) is approximately equal to

$$(1/4)^n \sqrt{\frac{4n^2}{\pi^2 n^4 - n^2(x^2+y^2)}} = (1/4)^n \frac{2}{\pi\sqrt{n^2-(x^2+y^2)}} \tag{13}$$

Note that this expression involve only the Euclidean form $x^2 + y^2$, which is rotationally invariant. (This is the first approximation involved in the proof.)

The exponent β is $nh(\frac{n+x+y}{2n}) + nh(\frac{n+x-y}{2n})$. Let $u = x/2n$ and $v = y/2n$. Then:

$$\beta = nh(\frac{1}{2} + u + v) + nh(\frac{1}{2} + u - v) \tag{14}$$

Since $u + v$ and $u - v$ are close to zero, we can approximate $h(1/2 + u + v)$ and $h(\frac{1}{2} + u - v)$ by expanding h around 1/2 up to the second order

$$h(\frac{1}{2} + p) = h(\frac{1}{2}) + h'(\frac{1}{2})p + 1/2h''(\frac{1}{2})p^2 \tag{15}$$

where

$$h(\frac{1}{2}) = -1/2 \ln 1/2 - 1/2 \ln 1/2 = \ln 2$$
$$h'(\frac{1}{2}) = -\ln 1/2 + \ln 1/2 = 0$$
$$h''(\frac{1}{2}) = -2 - 2 = 4$$

This is the second and last approximation involved in the proof. Plugging them into (14), we obtain:

$$\beta \approx n(\ln 2 - 2(u+v)^2) + n(\ln 2 - 2(u-v)^2) \tag{16}$$

$$= n \ln 4 - 4n(u^2+v^2) = n \ln 4 - \frac{x^2+y^2}{n} \tag{17}$$

Then, the exponential e^β is equal to $e^{n \ln 4 - \frac{x^2+y^2}{n}} = 4^n e^{(x^2+y^2)/n}$. Together with (13), we obtain that αe^β is equal to (9). □

B Quantum Mechanics

$\Psi(t) : \mathbb{Z}^n \to \mathbb{C}$ is the wavefunction of a certain quantum mechanical system at t in discrete space, with $\Psi(0)$ suitably spread out. $\tilde{\Psi}(t) : \mathbb{R}^n \to \mathbb{C}$ is its inverse Fourier series in the momentum space defined by $\tilde{\Psi}(t,p) = \sum_{x \in \mathbb{Z}^n} e^{-2\pi i p x} \Psi(t,x)$. In this appendix, I will call tiles "lattice points" which is more standard in this technical context.

Theorem B.1 *Let A be any rotation matrix in the momentum space. Assuming that the initial position space wavefunction $\Psi(0)$ is sufficiently spread out, we can show that the time evolution of $\tilde{\Psi}$ commutes with A. That is, $e^{-i\tilde{H}t}\tilde{\Psi}(0,Ap) \approx \tilde{\Psi}(t,Ap)$, where $e^{-i\tilde{H}t}$ is the operator that represents the time evolution up to t.*

Proof By applying the fourier series transform to $\hat{H}\Psi(t,x)$, we can get the momentum space Hamiltonian $\tilde{H}$ satisfying the Schrödinger equation $i\frac{d}{dt}\tilde{\Psi}(t) = \tilde{H}\tilde{\Psi}(t)$. We first observe that $\Psi(t, x+e_i)$ can be transformed to $e^{2\pi i p_i}\tilde{\Psi}(t,p)$ through the following steps (where $p_i = p \cdot e_i$, namely the i-th component of p):

$$\Psi(t, x+e_i) \Rightarrow \sum_{x\in\mathbb{Z}^n} e^{-2\pi ipx}\Psi(t, x+e_i) \tag{18}$$

$$= \sum_{x\in\mathbb{Z}^n} e^{-2\pi ip(x-e_i)}\Psi(t,x) \tag{19}$$

$$= \sum_{x\in\mathbb{Z}^n} e^{-2\pi ipx}e^{2\pi ipe_i}\Psi(t,x) \tag{20}$$

$$= e^{2\pi ip_i}\tilde{\Psi}(t,p) \tag{21}$$

Similarly, $\Psi(t, x-e_i)$ can be transformed to $e^{-2\pi ip_i}\tilde{\Psi}(t,p)$. Thus, from $\hat{H}\Psi(t,x) = -\frac{1}{2}\sum_i(\Psi(t,x+e_i)+\Psi(t,x-e_i)-2\Psi(t,x))$ we can obtain the following:

$$\tilde{H}\tilde{\Psi}(t,p) = -\frac{1}{2}\sum_i(e^{2\pi ip_i}+e^{-2\pi ip_i}-2)\tilde{\Psi}(t,p) \tag{22}$$

Let's abbreviate the expression "$-\frac{1}{2}\sum_i(e^{2\pi ip_i}+e^{-2\pi ip_i}-2)$" as "$\mathbf{Cow}(p)$". Then, $\tilde{H}\Psi(t,p) = \mathbf{Cow}(p)\Psi(t,p)$. Now we can solve the Schrödinger equation in the momentum space:

$$\tilde{\Psi}(t,p) = e^{-i\tilde{H}t}\tilde{\Psi}(0,p) = e^{-i\mathbf{Cow}(p)t}\tilde{\Psi}(0,p) \tag{23}$$

By assumption, the quantum mechanical system under consideration starts with a wavefunction sufficiently spread out around the origin at $t = 0$. That is, $\Psi(0,x)$ does not vary much over small distances. Given that $\tilde{\Psi}(0,p) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi ipx}\Psi(0,x)$, when p is large, $e^{-2\pi ipx}\Psi(0,x)$ tend to cancel off over small variation of x, so the results add up small ("destructive interference"). When p is very small (*i.e.*, $p \ll 1$), then there is no such destructive interference, and therefore the sum is much more significant. So $\tilde{\Psi}(0,p)$ is negligibly small for large values of p and only nonnegligible for small values of p.

We can show that $\mathbf{Cow}(p)$ is approximately rotationally invariant (a spherical cow) when p is sufficiently small:

$$\mathbf{Cow}(p) = -\frac{1}{2}\sum_i (e^{-2\pi i p_i} + e^{2\pi i p_i} - 2) \tag{24}$$

$$= -\sum_i (cos(2\pi p_i) - 1) \tag{25}$$

$$= -\sum_i (1 - 2\pi^2 p_i^2 - 1 + O(p_i^4)) \tag{26}$$

$$= 2\pi^2 \sum_i p_i^2 - O(p^4) \tag{27}$$

Here $\sum_i p_i^2$ is the square of the Euclidean length of p in the momentum space and is rotationally invariant. $O(p^4)$ is much smaller and can be ignored if $|p|$ is sufficiently small. Now if we look at the whole integral, we can see that large values of $|p|$ does not contribute much to the integral because $\tilde{\Psi}(0,p)$ is negligibly small for large $|p|$. So only small values of $|p|$ make main contribution to the integral. Since we can ignore $O(p^4)$ when $|p|$ is sufficiently small, then the whole integral is approximately invariant under any rotation of p in **Cow**.

Then, we obtain the desired result:

$$\tilde{\Psi}(t, Ap) = e^{-i\mathbf{Cow}(Ap)t}\tilde{\Psi}(0, Ap) \approx e^{-i\mathbf{Cow}(p)t}\tilde{\Psi}(0, Ap) = e^{-i\tilde{H}t}\tilde{\Psi}(0, Ap). \ \square$$

Theorem B.2 *If the time evolution of $\tilde{\Psi}$ is approximately isotropic, then the evolution of Ψ is also approximately isotropic: for any t and any rotation A acting on Ψ, $e^{-iHt}\Psi(0, Ax) \approx \Psi(t, Ax)$. (A rotation on discrete space can be considered an equivalence class of rotations on continuous space that map each point to places near the same lattice point.)*

Proof Let A be any rotation matrix on $\mathbb{R}^n$. We first extend $\Psi(t)$ to a continuous function $\Psi^+(t)$ over $\mathbb{R}^n$ with $\Psi^+(t,x) = \int_{p\in B} e^{2\pi ipx}\tilde{\Psi}(t,p)dp$. Then we can show that if $e^{-i\tilde{H}t}\tilde{\Psi}(0,Ap) \approx \tilde{\Psi}(t,Ap)$, then $e^{-i\hat{H}t}\Psi^+(0,Ax) \approx \Psi^+(t,Ax)$.

We have:

$$\Psi^+(t,Ax) = \int_{p\in B} e^{2\pi ipAx}\tilde{\Psi}(t,p)dp \tag{28}$$

$$= \int_{p\in AB} e^{2\pi iApAx}\tilde{\Psi}(t,Ap)dp \text{ (substitute } dp \text{ by } dA^{-1}p) \tag{29}$$

$$= \int_{p\in B} e^{2\pi ipx}\tilde{\Psi}(t,Ap)dp \ (ApAx = px) \tag{30}$$

$$\approx \int_{p\in B} e^{2\pi ipx}e^{-i\tilde{H}t}\tilde{\Psi}(0,Ap)dp \ (e^{-i\tilde{H}t}\tilde{\Psi}(0,Ap) \approx \tilde{\Psi}(t,Ap)) \tag{31}$$

$$= \int_{p\in B} e^{2\pi ipAx}e^{-i\tilde{H}t}\tilde{\Psi}(0,p)dp \text{ (similar to (29), (30))} \tag{32}$$

$$= e^{-i\hat{H}t}\Psi^+(0,Ax) \tag{33}$$

In (29), we substitute the integration variable p by $A^{-1}p$, so we integrate over the rotated Brillouin zone AB rather than B, and all the occurrences of p in (29) are replaced by Ap. To derive (30), we note that integrating over AB is about the same as integrating over B because only small values of p around the origin make main contributions to the integral (see B.1), which are all included in AB. Moreover, $ApAx = px$ because any rotation matrix is orthogonal, which means that it preserves the inner product of two vectors. From (32) to (33) we apply the fourier transform which commutes with the time evolution.

The result (33) means that the time evolution of the extended position wavefunction (over $\mathbb{R}^n$) is approximately isotropic. Of course, when x is a lattice point, Ax may not be. But Ax and its nearest lattice points have very similar Ψ-values because changing $|x|$ by one in the above equations make very little difference to the result (because the initial wavefunction is spread out by assumption, and only large x contribute significantly to the integral above). Therefore, we can conclude that the time evolution of the original position wavefunction is also approximately isotropic. Theorem B.2 follows. □

Theorem B.3 *For any quantum mechanical system with initial position $x \in \mathbb{Z}_n$, its position wavefunction will not evolve to be approximately rotationally invariant for any significant period of time: at any such duration, it is not the case that for any $y, z \in \mathbb{Z}_n$ that have approximately the same Euclidean distance to x, the amplitudes of the wavefunction at y and z are approximately the same.*

Let me first explain the qualification "any significant period of time." As we will see, the evolution of an arbitrary wavefunction can be rather erratic and it is possible for the wavefunction to be isotropic for a split-second, but this has no significance for our observations.

Proof. The main difference between what this theorem falsifies and Isotropy-qm is that the initial position in this theorem is a single lattice point rather than being spread out. Thus, all the reasoning in the case of Isotropy-qm that does not rely on the spread-out-ness of the initial position also applies here. What does not apply is the claim that the momentum wavefunction $\tilde{\Psi}(t)$ is approximately rotationally invariant (Theorem B.2). We can then translate this negative result into the position space and arrive at the conclusion that the position wavefunction is also not approximately rotationally invariant.

Recall that an important step for proving the approximate rotational invariance of $\tilde{\Psi}(t)$ is that $\tilde{\Psi}(0, p)$ is negligibly small for large values of p and only nonnegligible for small values of p. But this is not the case if $\Psi(0)$ is not spread out. Suppose $\Psi(0)$ is 1 at point zero and 0 elsewhere. In this case, we have $\tilde{\Psi}(0, p) = 1$. Then:

$$\Psi(t,x) = \int_{p\in B} e^{2\pi ipx}\tilde{\Psi}(t,p)dp \tag{34}$$
$$= \int_{p\in B} e^{2\pi ipx}e^{-i\mathbf{Cow}(p)t}\tilde{\Psi}(0,p)dp \tag{35}$$
$$= \int_{p\in B} e^{2\pi ipx}e^{-i\mathbf{Cow}(p)t}dp \tag{36}$$
$$= \int_{p\in B} e^{2\pi ipx}e^{-i(2\pi^2\sum_i p_i^2 - O(p^4))t}dp \tag{37}$$

Here, the influence of $O(p^4)$ is nonnegligible because the contribution of a neighborhood of large p (i.e., $|p|$ is close to $1/2$) to the integral is just as significant as a neighborhood of small p (i.e., $|p|$ is close to zero—far smaller than $1/2$) as far as the coefficient $e^{2\pi ipx}$ is concerned, so the "error" caused by $O(p^4)$ for large p cannot be suppressed by the coefficient. But how large is the deviation? For this question, it is helpful to provide some numerical analysis.

We can evaluate $\Psi(t)$ through the *modified Bessel function* $I(t)$ by the following equation (n is the dimension of space) (Olver and Maximon 2010; see Lemma B.4):

$$\Psi(t,x) = e^{-nit}\Pi_{j\leq n}I_{x_j}(it) \tag{38}$$

For simplicity, let's consider two-dimensional space, in which case we have

$$\Psi(t,x,y) = e^{-2it}I_x(it)I_y(it) \tag{39}$$

where $x,y \in \mathbb{Z}$ are the two space coordinates.

Then we can plot the modified Bessel function at a time t over regions where the amplitudes are significant. (Note that the "witnesses" to the isotropy violation are always regions where the amplitudes are significant, for if we measure the amplitudes far out in space where the values are negligibly low then ipso facto their differences are negligible.) As we can see in Figure 4, the wavefunctions are not approximately rotationally invariant. In each figure, there are obvious horizontal and vertical stripes representing different amplitudes from those of nearby points. These are deference patterns, which are responsible for the violation of isotropy, and such patterns will occur at all times (we can contrast this with Figure 2, where deference patterns are never as obvious and get more circle-like over time).

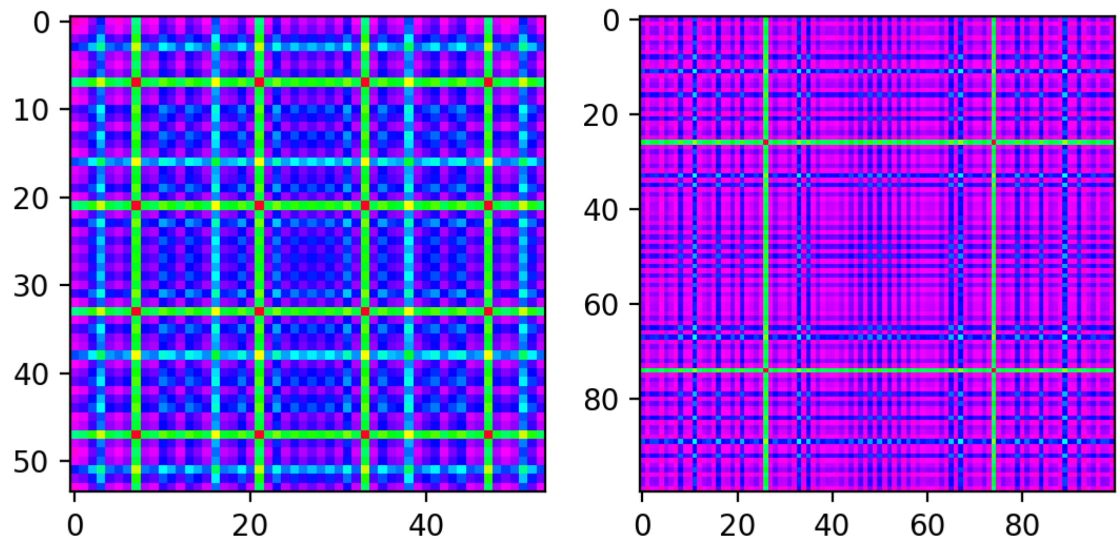

Figure 4: The time evolution of a quantum mechanical system with its initial position at a single lattice point. (Left) $t = 30$; (Right) $t = 100$. As before, the scale of each plot is adjusted so that we focus on the significant part of the wavefunction. The plait shirt pattern remains for later ts, with only the colors and frequencies of the stripes changing.

Lemma B.4 $\Psi(t,x) = e^{-nit}\Pi_{j\leq n} I_{x_j}(it)$ *is a solution to the Schrödinger equation* $\hat{H}\Psi(t,x) = -\frac{1}{2}\sum_i(\Psi(t,x+e_i)+\Psi(t,x-e_i)-2\Psi(t,x))$ *with the initial wavefunction* $\Psi(0)$ *being one at point zero and zero elsewhere.*

Proof For brevity I will prove the two-dimensional case $\Psi(t,x,y) = e^{-2it}I_x(it)I_y(it)$ (39), but the general n-dimensional case can be proved in the same way. In this case, $i\frac{\partial}{\partial t}\Psi(t,x,y) =$

$$\frac{1}{2}(4\Psi(t,x,y) - \Psi(t,x+1,y) - \Psi(t,x-1,y,) - \Psi(t,x,y+1) - \Psi(t,x,y-1)) \tag{40}$$

First, we check that the proposed solution (39) satisfies the initial condition. We can check that it follows from the definition of the modified Bessel function I that $I_x(0)$ is one at $x = 0$ and zero elsewhere (Olver and Maximon 2010, §10.25). So, at $t = 0$, $e^{-2it}I_x(it)I_y(it)$ is indeed equal to one at $x = y = 0$ and zero elsewhere.

We know that the modified Bessel function satisfies the following equation (ibid, §10.29):

$$2\frac{\partial I_x(it)}{\partial t} = iI_{x-1}(it) + iI_{x+1}(it) \tag{41}$$

By differentiating $\Psi(t,x,y) = e^{-2it}I_x(it)I_y(it)$ and making use of (41), we get:

$$i\frac{\partial \Psi(t,x,y)}{\partial t} = 2e^{-2it}I_x(it)I_y(it) + ie^{-2it}I_x(it)\frac{\partial I_y(it)}{\partial t} + ie^{-2it}\frac{\partial I_x(it)}{\partial t}I_y(it) \tag{42}$$

$$= 2\Psi(t,x,y) - \frac{1}{2}\Psi(t,x,y-1) - \frac{1}{2}\Psi(t,x,y+1) - \frac{1}{2}\Psi(t,x-1,y) - \frac{1}{2}\Psi(t,x+1,y) \tag{43}$$

(43) matches up with (40). Lemma B.4 follows. □

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